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Finite-temperature Néel transition in the CP^{N-1} model with one periodic spatial dimension

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Abstract. We investigate the finite-temperature Néel transition of the anisotropic large-N CP^{N-1} model with one periodic spatial dimension using the effective potential method. Long-range Néel order (LRNO) and the equation of the critical line are obtained. In $1+1$ dimensions, the one-loop approximation fails badly for large but finite N by the Mermin–Wagner theorem. But for infinite N, which is not realistic, it becomes exact and the appearance of LRNO is valid. The $(3+1)$-dimensional CP^{N-1} model is investigated as a model of antiferromagnetism in high-temperature superconductors, where the anisotropic parameter represents the weak interlayer coupling. The phase structure is very different from that of the $(1+1)$-dimensional infinite-N CP^{N-1} model.

1. Introduction

It is well known that the occurrence of high-temperature superconductivity is closely related to the spin degree of freedom of electrons in the Cu–O planes which can be described by the antiferromagnetic Heisenberg model. In the continuum limit this model is reduced to the CP^1 nonlinear σ model as a low-energy effective field theory [1]. Since we are concerned with the phase transition of Néel order which is a long-range phenomenon, it is convenient to consider the CP^1 model instead of the lattice Heisenberg model.

According to the Mermin–Wagner theorem [2], the Heisenberg model in space dimension $d \leqslant 2$ does not exhibit spontaneous breaking or the appearance of LRNO at any non-zero temperature. It has been shown that the Mermin–Wagner theorem holds for the large-N CP^{N-1} model in the infinite-size limit [3].

On the other hand, high-temperature superconductivity is characterized by a layered (or planar) structure [4] and has been thought to be described effectively by $(2+1)$-dimensional physics (e.g. the Chern–Simons theory). Thus if we regard high-temperature superconductivity as a $(2+1)$-dimensional phenomenon, it follows from the Mermin–Wagner theorem that high-temperature superconductors cannot have LRNO at any non-zero finite temperature. This contradicts the actual fact that some undoped materials have LRNO below some critical temperature T_N (Néel temperature). Therefore the appropriate model of magnetism in high-temperature superconductors is the $(3+1)$-dimensional anisotropic CP^1 model with weak interlayer coupling [3, 5]; we will not consider their detailed structures.

Recently the Gross–Neveu model and the CP^{N-1} model in $1+1$ dimensions were investigated on non-trivial topologies (i.e. on cylindrical and toroidal spacetimes) [6–8]. Their phase structures turned out to be rich. Physically toroidal spacetime $S^1 \times S^1$

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corresponds to finite temperature [9] and finite size [10] (more rigorously it corresponds to periodicity in space).

In the present paper we investigate the finite-temperature Néel transition of the $(3+1)$-dimensional anisotropic large-N CP^{N-1} model with weak interlayer coupling and periodicity in the third space dimension by making use of the effective-potential method [9, 11]. In addition, we consider the $(1+1)$-dimensional large-N CP^{N-1} model with spatial periodicity and examine the consistency with the Mermin–Wagner theorem. From the equilibrium condition of the effective potential we can find LRNO and the equation of the critical line (i.e. the phase diagram).

In section 2 we find the finite-temperature effective potential of the $(d+1)$-dimensional CP^{N-1} *model with one periodic spatial dimension, and its equilibrium condition. In* section 3 the $(1+1)$-dimensional CP^{N-1} model is considered. Contrary to the Mermin–Wagner–Coleman theorem [12], it has a region in the phase diagram where LRNO appears, which implies that the one-loop approximation fails badly for large but finite N. However, for infinite N, the one-loop approximation becomes exact [13] and the existence of LRNO is valid. In section 4, we consider the $(3+1)$-dimensional CP^{N-1} model as a model of magnetism in high-temperature superconductors. For the parameters of the real high-temperature superconductor La_2CuO_4, we obtain a phase diagram and LRNO which are very different from those of the $(1+1)$-dimensional infinite-N CP^{N-1} model. Section 5 contains a summary and our conclusions.

2. Effective potential and equilibrium condition in $d+1$ dimensions

We find the finite-temperature effective potential of the anisotropic large-N CP^{N-1} model with one periodic spatial dimension in $d+1$ dimensions. We work in Minkowski spacetime with the metric $g_{\mu\nu} = \mathrm{diag}(1, -1, \ldots, -1)$. For finite-temperature field theory we adopt the imaginary time formalism [9], where the time axis x^0 is compactified to a circle of circumference $(-\mathrm{i}\beta)$. In addition, for one periodic spatial dimension x^d, the x^d axis is also compactified to a circle of circumference L which is a periodic length.

Our model is given by the Lagrangian

$$\mathcal{L} = \sum_{\mu=0}^{d-1} \overline{D_\mu n} D^\mu n + \alpha \overline{D_d n} D^d n - \sigma\left(|n|^2 - \frac{N}{2f}\right) \tag{1}$$

where $n = (n_1(x), \ldots, n_N(x))$ and $D_\mu n = (\partial_\mu + \mathrm{i}A_\mu)n$. Here A_μ and σ are auxiliary fields. α is the ratio of the dth-space dimensional coupling constant f_d to the coupling constant f in the other spatial dimensions, that is, $\alpha = f_d/f$. α gives the anisotropy (or the interlayer coupling) to the CP^{N-1} model. Since the fields in Lagrangian (1) are bosonic, they are taken as periodic functions of x^0 and x^d, though the antiperiodic (or twisted) boundary condition can be given in a mathematical sense [14].

The partition function of Lagrangian (1) is given by

$$Z(\beta, L) = \int Dn\, D\overline{n}\, D\sigma\, DA_\mu \exp\left(\mathrm{i}\int_0^{-\mathrm{i}\beta} \mathrm{d}x^0 \int_0^L \mathrm{d}x^d \int \mathrm{d}^{d-1}x\, \mathcal{L}\right). \tag{2}$$

Integrating out the first $N-1$ components of n $(\overline{n})$ fields leaving only one component n_N $(\overline{n}_N)$ [3, 5, 10] in order to examine the appearance of LRNO, we obtain the effective action in terms of $n_N, \overline{n}_N$ and auxiliary fields

$$S_{\mathrm{eff}}(n_N, \overline{n}_N, \sigma, A_\mu) = N \int_0^{-\mathrm{i}\beta} \mathrm{d}x^0 \int_0^L \mathrm{d}x^d \int \mathrm{d}^{d-1}x \left[\sum_{\mu=0}^{d-1} \overline{D_\mu n_N} D^\mu n_N + \alpha \overline{D_d n_N} D^d n_N - \sigma \left(|n_N|^2 - \frac{1}{2f} \right) \right] + \mathrm{i}(N-1)\,\mathrm{Tr}\ln \left[\sum_{\mu=0}^{d-1} D_\mu D^\mu + \alpha D_d D^d + \sigma \right] \tag{3}$$

where we have rescaled $n_N \to \sqrt{N} n_N$ and $\overline{n}_N \to \sqrt{N}\overline{n}_N$.

Since we are concerned with the equilibrium state, we consider only the constant configurations of n_N ($\overline{n}_N$) and σ, and set $n_N = n_{N_c}$, $\overline{n}_N = \overline{n}_{N_c}$, $\sigma = \sigma_c$, $A_\mu = 0$. We now make some comments about the assumption $A_\mu = 0$: on spacetimes with non-trivial topology, any constant gauge field cannot globally be gauged to zero without altering the boundary condition of the quantum field along a non-contractible loop [8, 15, 16]. The non-zero expectation value of A_μ has an influence on the effective potential [15]. Our assumption $A_\mu = 0$ is a special case, which may be appropriate for the investigation of low-energy phenomena. In this case, Lagrangian (1) is reduced to the $O(2N)$ nonlinear σ model which describes the long-distance properties of the Heisenberg $O(2N)$ model [10].

For constant field configurations, we obtain

$$V_{\mathrm{eff}}(|n_{N_c}|, \sigma_c) = N\sigma_c \left(|n_{N_c}|^2 - \frac{1}{2f} \right) + \frac{N-1}{\sqrt{\alpha}} W_d(\sigma_c; \beta, L') \tag{4}$$

where

$$W_d(\sigma_c; \beta, L') \equiv \frac{1}{\beta L'} \sum_{m=-\infty}^{\infty}{}' \sum_{n=-\infty}^{\infty}{}' \int \frac{\mathrm{d}^{d-1}k}{(2\pi)^{d-1}} \ln \left[1 + \frac{\sigma_c}{k^2 + (2\pi m/\beta)^2 + (2\pi n/L')^2} \right] \tag{5}$$

and

$$L' \equiv \frac{L}{\sqrt{\alpha}}. \tag{6}$$

Here we used the fact that $k^0 = 2\pi m/(-\mathrm{i}\beta)$ and $k^d = 2\pi n/L$ (where m and n are integers) by the periodic boundary conditions. The primes on the summations denote the exclusion of the zero mode ($m = n = 0$). Note that W_d and V_{eff} are symmetric under the exchange of β and L'. If we rescale

$$V_{\mathrm{eff}} \to \frac{V_{\mathrm{eff}}}{\sqrt{\alpha}} \qquad |n_{N_c}|^2 \to \frac{|n_{N_c}|^2}{\sqrt{\alpha}} \qquad f \to \sqrt{\alpha} f \tag{7}$$

then equation (4) is reduced to

$$V_{\mathrm{eff}}(|n_{N_c}|, \sigma_c) = N\sigma_c\left(|n_{N_c}|^2 - 1/2f\right) + (N-1) W_d(\sigma_c; \beta, L'). \tag{8}$$

Since the quantum correction W_d is divergent, it needs to be renormalized.

Now we find the equilibrium conditions (or the stationary phase conditions) from the effective potential (8)

$$0=\frac{1}{N}\frac{\partial V_{\mathrm{eff}}}{\partial \overline{n}_{N_c}}=\sigma_c n_{N_c} \tag{9}$$

$$0=\frac{1}{N}\frac{\partial V_{\mathrm{eff}}}{\partial \sigma_c}=|n_{N_c}|^2-\frac{1}{2f} + \frac{N-1}{N\beta L'}\sum_m{}'\sum_n{}'\int\frac{\mathrm{d}^{d-1}k}{(2\pi)^{d-1}}\left[k^2+\left(\frac{2\pi m}{\beta}\right)^2+\left(\frac{2\pi n}{L'}\right)^2+\sigma_c\right]^{-1}. \tag{10}$$

Since we are concerned with the existence of LRNO (i.e. $n_{N_c}\neq 0$), $\sigma_c=0$ from (9). Then LRNO n_{N_c} is obtained from (10) after some renormalization.

3. Néel transition in $1+1$ dimensions

In $1+1$ dimensions, the appearance of LRNO is forbidden by the Mermin–Wagner–Coleman theorem. For the $(1+1)$-dimensional large-N CP^{N-1} model, we examine whether or not this is the case in the one-loop approximation.

In order to find the effective potential (8), we evaluate $W_1(\sigma_c;\beta,L)$. In $1+1$ dimensions we take $\alpha=1$, since there is only one spatial dimension and thus the anisotropy of space is meaningless

$$\begin{aligned}W_1(\sigma_c;\beta,L)&=\frac{1}{\beta L}\sum_m{}'\sum_n{}'\ln\left[1+\frac{\sigma_c}{(2\pi m/\beta)^2+(2\pi n/L)^2}\right]\\&=\frac{1}{\beta L}\sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty}\ln\left[1+\frac{\sigma_c}{(2\pi m/\beta)^2+(2\pi n/L)^2+\mathrm{i}\varepsilon}\right]-\frac{1}{\beta L}\ln\left(1+\frac{\sigma_c}{\mathrm{i}\varepsilon}\right).\end{aligned} \tag{11}$$

By successive applications of the summation formula

$$\sum_{n=-\infty}^{\infty}\ln\left[1+\frac{b^2}{a^2+n^2}\right]=\int_{-\infty}^{\infty}\mathrm{d}x\,\ln\left[1+\frac{b^2}{a^2+x^2}\right]+2\ln\left[\frac{1-\exp(-2\pi\sqrt{a^2+b^2})}{1-\exp(-2\pi|a|)}\right] \tag{12}$$

we can find

$$W_1(\sigma_c;\beta,L)=W_1^{R^2}(\sigma_c)+\Delta W_1(\sigma_c;\beta,L) \tag{13}$$

where $W_1^{R^2}(\sigma_c)$ is the one-loop contribution to the effective potential in R^2 and $\Delta W_1(\sigma_c;\beta,L)$ is the effect of the change of topology (i.e. finite β and L)

$$W_1^{R^2}(\sigma_c)=\int\frac{\mathrm{d}^2k_E}{(2\pi)^2}\ln\left(1+\frac{\sigma_c}{k_E^2}\right) \tag{14}$$

$$\begin{aligned}\Delta W_1(\sigma_c;\beta,L)&=\frac{2}{\pi\beta^2}\int_0^{\infty}\mathrm{d}x\,\ln\left[\frac{1-\exp(-\sqrt{x^2+\beta^2\sigma_c})}{1-\exp(-x)}\right]\\&\quad+\frac{4}{\beta L}\sum_{n=1}^{\infty}\ln\left[\frac{1-\exp\left(-\sqrt{(2\pi Ln/\beta)^2+L^2\sigma_c}\right)}{1-\exp(-2\pi Ln/\beta)}\right]\\&\quad+\frac{2}{\beta L}\ln\left[\frac{1-\exp(-L\sqrt{\sigma_c})}{L\sqrt{\sigma_c}}\right].\end{aligned} \tag{15}$$

Note that $\Delta W_1(\sigma_c; \beta \to \infty, L \to \infty) = 0$. $\Delta W_1(\sigma_c; \beta, L)$ has a hidden exchange symmetry between β and L. Though we can write $\Delta W_1(\sigma_c; \beta, L)$ in an explicitly symmetrized form with respect to β and L (that is, $\Delta W_1(\sigma_c; \beta, L) = \frac{1}{2}[\Delta W_1(\sigma_c; \beta, L) + \Delta W_1(\sigma_c; L, \beta)]$), we prefer the unsymmetrized expression (15) for later convenience.

Now we define some useful functions

$$g(z) \equiv \int_0^\infty \mathrm{d}x \, \ln\left[\frac{1-\exp(-\sqrt{x^2+z^2})}{1-\exp(-x)}\right] = \frac{\pi^2}{6} - \int_z^\infty \mathrm{d}x \, \frac{\sqrt{x^2-z^2}}{\mathrm{e}^x - 1} \tag{16}$$

$$s(z; \beta, L) \equiv \sum_{n=1}^\infty \ln\left[\frac{1-\exp\left(-\sqrt{(2\pi Ln/\beta)^2+z^2}\right)}{1-\exp(-2\pi Ln/\beta)}\right] \tag{17}$$

$$h(z) \equiv \ln\left(\frac{1-\mathrm{e}^{-z}}{z}\right). \tag{18}$$

Since $g(z)$ is manifestly finite, the summations in $\Delta W_1(\sigma_c; \beta, L)$ are also finite, thus ΔW_1 is finite but $W_1^{R^2}$ is divergent. Therefore in order to renormalize V_{eff}, it is sufficient to renormalize the effective potential $V_{\mathrm{eff}}^{R^2}$ in R^2, where

$$\begin{aligned} V_{\mathrm{eff}}^{R^2}(|n_{N_c}|, \sigma_c) &= V_{\mathrm{eff}}(|n_{N_c}|, \sigma_c; \beta \to \infty, L \to \infty) \\ &= N\sigma_c\left(|n_{N_c}|^2 - 1/2f\right) + (N-1)W_1^{R^2}(\sigma_c) \\ &= N\sigma_c\left(|n_{N_c}|^2 - \frac{1}{2f}\right) - \frac{N-1}{4\pi}\sigma_c\left(\ln\frac{\sigma_c}{\Lambda^2} - 1\right) \end{aligned} \tag{19}$$

in the infinite momentum cutoff limit ($\Lambda \to \infty$). The cutoff dependence can be absorbed in the renormalization of the coupling constant f. If we take the renormalization condition as

$$\frac{1}{N}\frac{\partial V_{\mathrm{eff}}^{R^2}}{\partial \sigma_c}\bigg|_{n_{N_c}=0, \sigma_c=M^2} = -\frac{1}{2f_r} \tag{20}$$

where M is the renormalization point and f_r the renormalized coupling constant, we obtain the renormalized effective potential in R^2

$$\begin{aligned} V_r^{R^2}(|n_{N_c}|, \sigma_c) &= N\sigma_c\left(|n_{N_c}|^2 - \frac{1}{2f_r}\right) - \frac{N-1}{4\pi}\sigma_c\left(\ln\frac{\sigma_c}{M^2} - 1\right) \\ &= N\sigma_c|n_{N_c}|^2 - \frac{N-1}{4\pi}\sigma_c\left(\ln\frac{\sigma_c}{\sigma_0} - 1\right) \end{aligned} \tag{21}$$

with

$$\sigma_0 \equiv M^2 \exp\left[-\frac{2\pi N}{(N-1)f_r}\right]. \tag{22}$$

Note that f_r was absorbed into σ_0. Finally we find the renormalized effective potential V_r at finite temperature and finite size

$$\begin{aligned}V_r(|n_{N_c}|, \sigma_c; \beta, L) &= V_r^{R^2}(|n_{N_c}|, \sigma_c) + (N-1)\Delta W_1(\sigma_c; \beta, L)\\ &= N\sigma_c|n_{N_c}|^2 - \frac{N-1}{4\pi}\sigma_c\left(\ln\frac{\sigma_c}{\sigma_0} - 1\right)\\ &\quad + 2(N-1)\left[\frac{g(\beta\sqrt{\sigma_c})}{\pi\beta^2} + \frac{2}{\beta L}s(L\sqrt{\sigma_c}; \beta, L) + \frac{h(L\sqrt{\sigma_c})}{\beta L}\right]. \end{aligned} \tag{23}$$

Using this renormalized effective potential V_r, we find the equilibrium condition (10)

$$\begin{aligned}0 = \frac{1}{N}\frac{\partial V_r}{\partial \sigma_c} &= |n_{N_c}|^2 - \frac{N-1}{4\pi N}\ln\frac{\sigma_c}{\sigma_0}\\ &\quad + \frac{2(N-1)}{N}\left[\frac{1}{2\pi}\int_0^\infty \mathrm{d}x\, \frac{1}{\sqrt{x^2+\beta^2\sigma_c}}\frac{1}{\exp\{\sqrt{x^2+\beta^2\sigma_c}\}-1}\right.\\ &\quad + \frac{L}{\beta}\sum_{n=1}^{\infty}\left\{\sqrt{(2\pi Ln/\beta)^2 + L^2\sigma_c}\left(\exp\{\sqrt{(2\pi Ln/\beta)^2+L^2\sigma_c}\}-1\right)\right\}^{-1}\\ &\quad \left. + \frac{1}{2\beta L\sigma_c}\left(\frac{L\sqrt{\sigma_c}}{\exp\{L\sqrt{\sigma_c}\}-1} - 1\right)\right] \end{aligned} \tag{24}$$

$$\begin{aligned} &= |n_{N_c}|^2 + \frac{N-1}{2\pi N}\left[\ln\left(\frac{\beta\sqrt{\sigma_0}}{4\pi}\right) + \gamma + \frac{\pi}{6}\frac{L}{\beta}\right.\\ &\quad \left. + 2\sum_{n=1}^{\infty}\frac{1}{n}\frac{1}{\exp\{2\pi Ln/\beta\}-1}\right] + \mathrm{O}(\sqrt{\sigma_c}) \end{aligned} \tag{25}$$

where for the last identity we expanded (24) for small σ_c with the help of the integration formula [9]

$$\int_0^\infty \mathrm{d}x\, \frac{x^{-\varepsilon}}{\sqrt{x^2+a^2}}\frac{1}{\exp\{\sqrt{x^2+a^2}\}-1} = \frac{\pi}{2a} + \frac{1}{2}\ln\frac{a}{4\pi} + \frac{\gamma}{2} + \mathrm{O}(\varepsilon) + \mathrm{O}(a^2) \tag{26}$$

for small ε and a^2. Here $\gamma(= 0.577\ldots)$ is the Euler constant. Since we are interested in the appearance of LRNO (i.e. $n_{N_c} \neq 0$), $\sigma_c = 0$ from (9). Setting $\sigma_c = 0$ in (25) and using the identity [7]

$$\sum_{n=1}^{\infty}\frac{1}{n}\frac{1}{\exp\{2\pi Ln/\beta\}-1} = -\frac{\pi}{12}\frac{L}{\beta} - \ln\left[\eta\left(\mathrm{i}\frac{L}{\beta}\right)\right] \tag{27}$$

and LRNO is given by

$$|\tilde{n}_{N_c}|^2 = \frac{1}{2\pi}\left[2\ln\left\{\eta\left(\mathrm{i}\frac{\tilde{L}}{\tilde{\beta}}\right)\right\} - \ln\left(\frac{\tilde{\beta}}{4\pi}\right) - \gamma\right] \tag{28}$$

where η is the celebrated Dedekind η function [17] which is defined as

$$\eta(z) \equiv q^{1/24}\prod_{n=1}^{\infty}(1-q^n) \tag{29}$$

with $q(z) = \exp(2\pi i z)$. Here we introduced the dimensionless quantities

$$|\tilde{n}_{N_c}|^2 = \frac{N}{N-1}|n_{N_c}|^2 \qquad \tilde{\beta} = \sqrt{\sigma_0}\beta \qquad \tilde{L} = \sqrt{\sigma_0}L. \tag{30}$$

In (28), $|\tilde{n}_{N_c}|^2$ must be positive definite in order to be physically meaningful, otherwise it is set to be zero. The equation of the critical line is obtained from (28) by setting $\tilde{n}_{N_c} = 0$: exchanging $\tilde{\beta}$ and $\tilde{L}$ for convenience

$$\tilde{L} = 4\pi e^{-\gamma}[\eta(i\tilde{\beta}/\tilde{L})]^2. \tag{31}$$

Note that both LRNO (28) and the equation of the critical line (31) have a hidden exchange symmetry between $\tilde{\beta}$ and $\tilde{L}$ originating from the same symmetry of the effective potential.

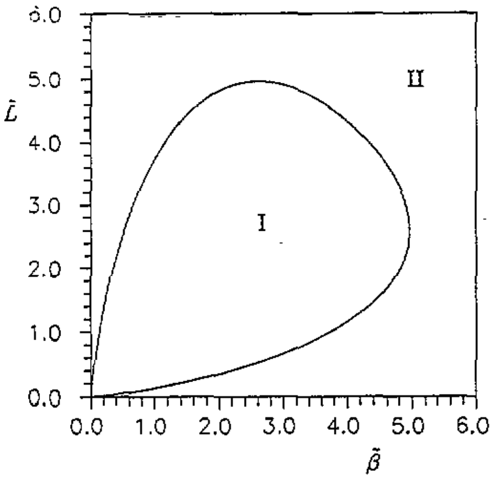

Figure 1. The phase diagram in the $\tilde{\beta}$ and $\tilde{L}$ space. LRNO appears in region I but not in region II. For $\tilde{L} \geqslant \tilde{L}_0(\approx 4.99)$, the system has no critical temperature (no phase transition) while it has two critical temperatures (two phase transitions) for $0 < \tilde{L} < \tilde{L}_0$.

Figure 1 shows the phase diagram (or the critical line). LRNO appears in region I of the phase diagram but not in region II. The existence of LRNO contradicts the Mermin–Wagner–Coleman theorem. Hence this implies that the one-loop (or large-N) approximation fails badly for large but finite N. However, for the model with infinite N which is not a real physical system, the one-loop approximation is exact and our results are valid; that is, LRNO can exist. In the rest of this section, we will restrict ourselves to the infinite-N case. On the other hand, if we see the phase in terms of the spin mass gap σ_c, the opposite situation occurs by (9). That is, there is dynamical mass generation in region II (i.e. $\sigma_c \neq 0$) but not in region I (i.e. $\sigma_c = 0$) [8]. In the phase diagram there exists $\tilde{L}_0$ such that the system has two critical temperatures for fixed $\tilde{L}$ satisfying $0 < \tilde{L} < \tilde{L}_0$, while it has no critical temperature for $\tilde{L} \geqslant \tilde{L}_0$

$$\tilde{L}_0 = 4\pi e^{-\gamma}[\eta(iz_0)]^2 \approx 4.99$$

where $z_0(\approx 0.5235)$ is the value such that $\eta'(iz_0) = 0$. $\eta(iz_0)(\approx 0.8382)$ is the maximum of $\eta(iz)$. By the symmetry between $\tilde{\beta}$ and $\tilde{L}$, the same holds under the exchange of $\tilde{\beta}$ and $\tilde{L}$.

Figure 2 shows LRNO $\tilde{n}_{N_c}$ (28) as a function of $\tilde{\beta}$ and $\tilde{L}$. Figure 3 is the cross sections of figure 2 as functions of $\tilde{\beta}$ for various fixed values of $\tilde{L} < \tilde{L}_0$. As expected from figure 1, there exist two critical temperatures such that LRNO appears between them.

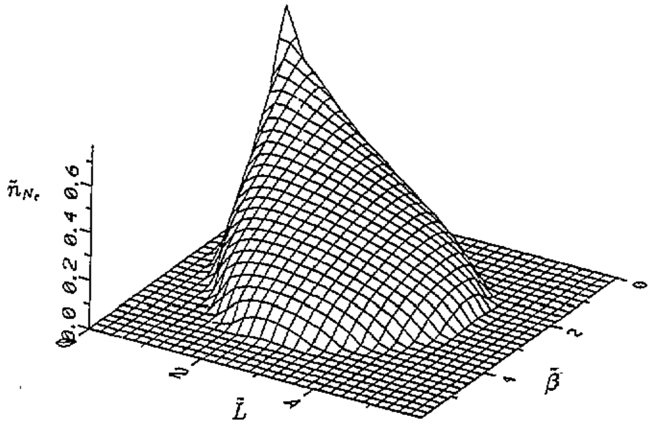

Figure 2. The three-dimensional plot of LRNO $\tilde{n}_{N_c}$ as a function of $\tilde{\beta}$ and $\tilde{L}$.

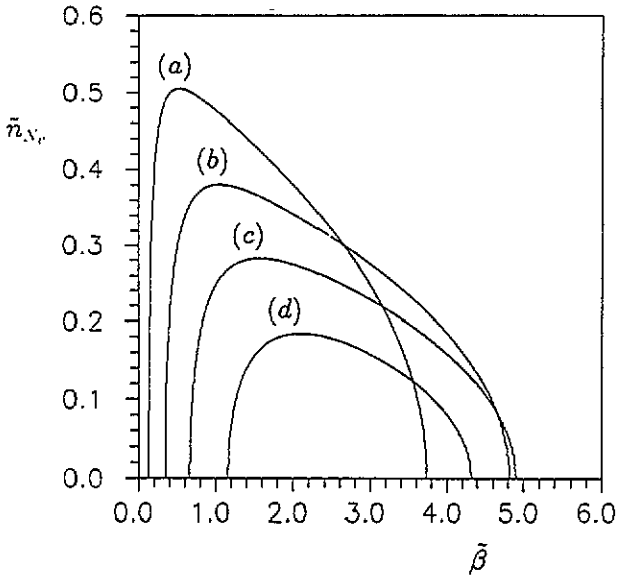

Figure 3. LRNO $\tilde{n}_{N_c}$ as a function of $\tilde{\beta}$ for various values of $\tilde{L} < \tilde{L}_0$; that is, $\tilde{L} = 1, 2, 3, 4$ for curves (*a*)–(*d*), respectively. As expected from figure 1, these cases have two critical temperatures between which LRNO exists.

4. Néel transition in $3 + 1$ dimensions

In this section we consider the effect of finite temperature and finite size (or periodicity) in the third space dimension orthogonal to the planes on the Néel transition of the anisotropic CP^{N-1} model in $3+1$ dimensions, which is regarded as a model of the magnetism in high-temperature superconductors. In $3+1$ dimensions the appearance of LRNO is not forbidden by the Mermin–Wagner theorem.

From (8), the effective potential is given by

$$V_{\rm eff}(|n_{N_c}|,\sigma_c) = N\sigma_c\big(|n_{N_c}|^2 - 1/2f\big) + (N-1)W_3(\sigma_c;\beta,L') \tag{32}$$

where

$$W_3(\sigma_c;\beta,L') = \frac{1}{\beta L'}\sum_m{}'\sum_n{}'\int\frac{\mathrm{d}^2k}{(2\pi)^2}\ln\left[1+\frac{\sigma_c}{k^2+(2\pi m/\beta)^2+(2\pi n/L')^2}\right] \tag{33}$$

with $L' = L/\sqrt{\alpha}$. We write W_3 as $W_3 = W_3^{R^4} + \Delta W_3$ where $W_3^{R^4}$ is the one-loop contribution to the effective potential in R^4 and ΔW_3 is the effect of the change of topology (i.e. finite β and L'). The cutoff dependence of $W_3^{R^4}$, which is manifestly divergent in the infinite cutoff limit, cannot be absorbed into only one coupling constant renormalization since $W_3^{R^4}$ has two kinds of infinite terms [9]. Hence $V_{\rm eff}$ is not renormalizable in the infinite momentum cutoff limit.

However, our interest is a physical quantity measurable by experiments, that is, LRNO n_{N_c} which is obtained from the equilibrium conditions. We will find LRNO with finite momentum cutoffs related to the lattice constants. The equilibrium condition (10) is

$$0 = \frac{1}{N}\frac{\partial V_{\rm eff}}{\partial\sigma_c} = |n_{N_c}|^2 - \frac{1}{2f} + \frac{N-1}{N}\frac{\partial W_3}{\partial\sigma_c} \tag{34}$$

where

$$\frac{\partial W_3(\sigma_c;\beta,L')}{\partial\sigma_c} = \frac{1}{\beta L'}\sum_m{}'\sum_n{}'\int\frac{\mathrm{d}^2k}{(2\pi)^2}\left[k^2+\left(\frac{2\pi m}{\beta}\right)^2+\left(\frac{2\pi n}{L'}\right)^2+\sigma_c\right]^{-1}. \tag{35}$$

In order to regularize (35) and renormalize (34) we introduce the finite momentum cutoffs [3, 5]: Λ is a two-dimensional momentum cutoff determined from the two-dimensional lattice constant a by the relation

$$\Lambda a = \sqrt{2\pi} \tag{36}$$

which conserves the area of the Brillouin zone for the antiferromagnetically ordered state in the Cu–O plane. In addition, Λ_3 is the third space momentum cutoff which is related to the third space lattice constant a_3 by

$$\Lambda_3 a_3 = \pi \tag{37}$$

then

$$\frac{\partial W_3}{\partial\sigma_c} = \frac{1}{4\pi\beta L'}\sum_m{}'\sum_n{}'\ln\left[1+\frac{\Lambda^2}{(2\pi m/\beta)^2+(2\pi n/L')^2+\sigma_c}\right].$$

We restrict ourselves to the $\sigma_c = 0$ phase where LRNO appears

$$\begin{aligned}4\pi\left.\frac{\partial W_3}{\partial\sigma_c}\right|_{\sigma_c=0} &= W_1(\Lambda^2;\beta,L') \\ &= W_1^{R^2}(\Lambda^2) + \Delta W_1(\Lambda^2;\beta,L'). \end{aligned} \tag{38}$$

Here

$$W_1^{R^2}(\Lambda^2) = \int \frac{d^2k_E}{(2\pi)^2} \ln\left(1 + \frac{\Lambda^2}{k_E^2}\right)$$
$$= \frac{1}{(2\pi)^2} \int_{-\sqrt{\alpha}\Lambda_3}^{\sqrt{\alpha}\Lambda_3} dk_3 \int_{-\infty}^{\infty} dk_4 \ln\left(1 + \frac{\Lambda^2}{k_3^2 + k_4^2}\right) \tag{39}$$
$$= \frac{\Lambda^2}{2\pi} Q(\alpha\delta) \tag{40}$$

where

$$Q(z) \equiv \sqrt{z}\left(\sqrt{z+1} - \sqrt{z}\right) + \ln\left(\sqrt{z+1} + \sqrt{z}\right) \tag{41}$$

and

$$\Lambda_3 \equiv \sqrt{\delta}\Lambda \qquad \sqrt{\delta} \sim O(1). \tag{42}$$

Equation (40) was also obtained in [3] with a slight difference by a factor $\sqrt{\alpha}$ due to the rescaling (7). Note that in (39) the momentum cutoff of the k_3 integral was changed from Λ_3 to $\sqrt{\alpha}\Lambda_3$ because of the anisotropic parameter α, and that of the k_4 integral is infinite because the Euclidean time axis is continuous without the lattice spacing.

Now we take the renormalization condition as

$$\frac{1}{N}\frac{\partial V_{\text{eff}}^{R^4}}{\partial \sigma_c}\bigg|_{n_{N_c}=\sigma_c=0} = \frac{1}{N}\frac{\partial V_{\text{eff}}}{\partial \sigma_c}\bigg|_{\substack{n_{N_c}=\sigma_c=0\\ \beta=L\to\infty}}$$
$$= -\frac{1}{2f} + \frac{N-1}{4\pi N} W_1^{R^2}(\Lambda^2) = -\frac{1}{2f_r} \tag{43}$$

where (38) was used for the second identity and $W_1^{R^2}(\Lambda^2)$ is given by (40). Then LRNO can be obtained from (34) with $\sigma_c = 0$, using (38) and (43)

$$|\tilde{n}_{N_c}|^2 = \frac{1}{2\tilde{f}_r} - \frac{1}{2\pi}\left[\frac{g(\tilde{\beta})}{\pi\tilde{\beta}^2} + \frac{2}{\tilde{\beta}\tilde{L}'} s(\tilde{L}'; \tilde{\beta}, \tilde{L}') + \frac{h(\tilde{L}')}{\tilde{\beta}\tilde{L}'}\right] \tag{44}$$

where we have introduced the dimensionless quantities

$$|\tilde{n}_{N_c}|^2 = \frac{N}{N-1}\frac{|n_{N_c}|^2}{\Lambda^2} \qquad \tilde{f}_r = \frac{N-1}{N} f_r \Lambda^2 \qquad \tilde{\beta} = \beta\Lambda \qquad \tilde{L}' = L'\Lambda \tag{45}$$

and the functions g, s and h were defined in (16)–(18), respectively. The equation of the critical line is obtained by setting $\tilde{n}_{N_c} = 0$ in (44)

$$\frac{g(\tilde{\beta})}{\pi\tilde{\beta}^2} + \frac{2}{\tilde{\beta}\tilde{L}'} s(\tilde{L}'; \tilde{\beta}, \tilde{L}') + \frac{h(\tilde{L}')}{\tilde{\beta}\tilde{L}'} = \frac{\pi}{\tilde{f}_r}. \tag{46}$$

Since LRNO and the equation of the critical line have a hidden exchange symmetry between $\tilde{\beta}$ and $\tilde{L}'$, it is sufficient to consider only the $\tilde{L}' \geqslant \tilde{\beta}$ region. It is convenient to use (44) to investigate this region, which is the reason why we preferred the unsymmetrized expression of ΔW_1 (15); for $\tilde{L}' \geqslant \tilde{\beta}$, LRNO (44) can be approximated by

$$|\tilde{n}_{N_c}|^2 = \frac{1}{2\tilde{f}_r} - \frac{1}{2\pi}\left[\frac{g(\tilde{\beta})}{\pi\tilde{\beta}^2} + \frac{2}{\tilde{\beta}\tilde{L}'}\ln\left(\frac{1-\exp\left(-\sqrt{\tilde{L}'^2+(2\pi\rho)^2}\right)}{1-\exp(-2\pi\rho)}\right) + \frac{h(\tilde{L}')}{\tilde{\beta}\tilde{L}'}\right] \tag{47}$$

where $\rho \equiv \tilde{L}'/\tilde{\beta}$. In order to find more simple expressions for LRNO and the critical line, consider the limiting cases for $\tilde{\beta}$ and $\tilde{L}'$.

(i) The case of low temperature and large size (i.e. large $\tilde{\beta}$ and $\tilde{L}'$)
For $\tilde{L}' \geqslant \tilde{\beta} \gg 1$

$$|\tilde{n}_{N_c}|^2 = \frac{1}{2}\left(\frac{1}{\tilde{f}_r} - \frac{g(\tilde{\beta})}{(\pi\tilde{\beta})^2} + \frac{\ln\tilde{L}'}{\pi\tilde{\beta}\tilde{L}'}\right) \tag{48}$$

where we neglected the second term in the bracket of (47). LRNO for $\tilde{\beta} \geqslant \tilde{L}' \gg 1$ is obtained by exchanging $\tilde{\beta}$ and $\tilde{L}'$ in (48).

Now we consider the infinite-size limit ($\tilde{L}' \to \infty$) at low temperature using (48)

$$\begin{aligned}|\tilde{n}_{N_c}|^2 &= \frac{1}{2}\left(\frac{1}{\tilde{f}_r} - \frac{g(\tilde{\beta})}{(\pi\tilde{\beta})^2}\right)\\ &\approx \frac{1}{2}\left(\frac{1}{\tilde{f}_r} - \frac{1}{6\tilde{\beta}^2}\right).\end{aligned} \tag{49}$$

Here we used the asymptotic expansion of $g(\tilde{\beta})$, for large $\tilde{\beta}$

$$\begin{aligned}g(\tilde{\beta}) &\approx \frac{\pi^2}{6} - \tilde{\beta}K_1(\tilde{\beta})\\ &= \frac{\pi^2}{6} - \sqrt{\frac{\pi\tilde{\beta}}{2}}e^{-\tilde{\beta}} + \cdots\end{aligned} \tag{50}$$

where $K_1(\tilde{\beta})$ is a modified Bessel function of order one. According to the experimental evidence, assume that there exists LRNO at zero temperature ($\tilde{\beta} \to \infty$). Then $\tilde{f}_r > 0$ and from (43) we obtain the condition which the coupling constant $\tilde{f}(\equiv (N-1/N)f\Lambda^2)$ must satisfy

$$\tilde{f} < \frac{(2\pi)^2}{Q(\alpha\delta)} \equiv \tilde{f}_c \tag{51}$$

in terms of $\tilde{f}_c$

$$\frac{1}{\tilde{f}_r} = \frac{1}{\tilde{f}} - \frac{1}{\tilde{f}_c}. \tag{52}$$

To be physically meaningful $|\tilde{n}_{N_c}|^2$ must be positive definite in (49) so that

$$\tilde{T} \leqslant \sqrt{6/\tilde{f}_r} \equiv \tilde{T}_N \tag{53}$$

where $\tilde{T} \equiv \tilde{\beta}^{-1}$ and $\tilde{T}_{\rm N}$ is the Néel temperature in the infinite-size limit. Considering the rescaling of the coupling constant f (7) which was performed earlier, it can be shown that $\tilde{T}_{\rm N} \propto \alpha^{1/4}$ for small α. Thus when $\alpha = 0$ (i.e. in $2+1$ dimensions), $\tilde{T}_{\rm N} = 0$, which is consistent with the Mermin–Wagner theorem. In terms of $\tilde{T}_{\rm N}$, equation (49) can be rewritten as

$$|\tilde{n}_{N_c}|^2 = \tfrac{1}{12}(\tilde{T}_{\rm N}^2 - \tilde{T}^2). \tag{54}$$

In the infinite-size limit, LRNO appears if and only if $\tilde{T} < \tilde{T}_{\rm N}$ or $\tilde{\beta} > \tilde{\beta}_{\rm N}$. Alternatively, at zero temperature, LRNO exists if and only if $\tilde{L}' > \tilde{L}'_{\rm N}$ $(= \tilde{\beta}_{\rm N})$, where $\tilde{L}'_{\rm N}$ may be called the Néel periodic length.

At this point we give a numerical estimation of physical parameters using the experimental results on high-temperature superconductors [3,5]. We take La_2CuO_4 as a sample. Then we have $a = 3.8$ Å, the shortest Cu–Cu distance in the Cu–O planes and $a_3 = 6.6$ Å, the distance between the planes, and the anisotropic parameter $\alpha \approx 10^{-5}$. Using (36), (37) and (42), $\delta \approx 0.52$. Then $\tilde{f}_{\rm c} \approx 8660$ by (51). Using (53) $\tilde{f}_{\rm r}$ can be determined from the Néel temperature in the infinite-size limit, $T_{\rm N} \approx 200$ K. Since 1 Å$^{-1} \approx 2.285 \times 10^7$ K, $\Lambda \approx 1.5 \times 10^7$ K using (36). Then $\tilde{T}_{\rm N} = T_{\rm N}/\Lambda \approx 1.3 \times 10^{-5} (\ll 1)$, which makes our low-temperature expansion reasonable. Equation (53) gives $\tilde{f}_{\rm r} \approx 3.4 \times 10^{10}$. $\tilde{\beta} = 1$ and $\tilde{L}' = 1$ correspond to $T = \Lambda \approx 1.5 \times 10^7$ K and $L' = \Lambda^{-1} \approx 1.5$ Å (or $L \approx 0.0048$ Å $\ll$ lattice constant!), respectively. It must be emphasized that our theoretical investigation will be connected with the magnetic property of the real high-temperature superconductor La_2CuO_4 only at relatively low temperature (e.g. below the melting temperature of the lattice structure). Our results for very high temperature or very small periodic length must be regarded as purely theoretical results of the CP^{N-1} model which are independent of real materials.

Now we find the equation of the critical line: for $\tilde{L}' \geqslant \tilde{\beta} \gg 1$, using (48)

$$\begin{aligned} \frac{\ln \tilde{L}'}{\tilde{L}'} &= -\pi\tilde{\beta}\left(\frac{1}{\tilde{f}_{\rm r}} - \frac{g(\tilde{\beta})}{(\pi\tilde{\beta})^2}\right) \\ &\approx -\pi\tilde{\beta}\left(\frac{1}{\tilde{f}_{\rm r}} - \frac{1}{6\tilde{\beta}^2}\right). \end{aligned} \tag{55}$$

This predicts the existence of the asymptotic line, $\tilde{\beta} \to \tilde{\beta}_{\rm N}$ as $\tilde{L}' \to \infty$. In addition, the equation of the critical line for $\tilde{\beta} \geqslant \tilde{L}' \gg 1$ is obtained by exchanging $\tilde{\beta}$ and $\tilde{L}'$ in (55).

(ii) The case of high temperature and small size (i.e. small $\tilde{\beta}$ and $\tilde{L}'$)

For $\tilde{\beta}$ and $\tilde{L}' \ll 1$, the system has much higher temperature than the momentum cutoff and *much smaller size (or periodic length) than the lattice constant in the third space dimension.* Hence this case can exist only as a very unstable state.

Using (A6), we find LRNO for small $\tilde{\beta}$ and $\tilde{L}'$

$$|\tilde{n}_{N_c}|^2 = \frac{1}{2\tilde{f}_{\rm r}} + \frac{1}{8\pi^2}\left(2\ln\eta({\rm i}\rho) - \ln\frac{\tilde{\beta}}{4\pi} - \gamma + \frac{1}{2}\right) \tag{56}$$

up to order O (1). The terms neglected here are much less than the O (1) term in the region where LRNO appears, which will be justified soon. Equation (56) has the same form as LRNO (28) in $1+1$ dimensions and has a hidden exchange symmetry between $\tilde{\beta}$ and $\tilde{L}'$.

As in 1 + 1 dimensions (see (21), (22) and (30)), $\tilde{f}_r$ can be absorbed in the rescaling of $\tilde{\beta}$ and $\tilde{L}'$.

Now we obtain the equation of the critical line by setting $\tilde{n}_{N_c} = 0$ in (56): for small $\tilde{\beta}$ and $\tilde{L}'$

$$\tilde{\beta} = 4\pi \exp\left(-\gamma + \frac{1}{2} + \frac{4\pi^2}{\tilde{f}_r} \right) [\eta(i\rho)]^2. \tag{57}$$

To express this in more familiar form, consider the $\tilde{L}' \geqslant \tilde{\beta}$ region. Using (29), the upper (i.e. $\tilde{L}' \geqslant \tilde{\beta}$) critical line is approximated by

$$\tilde{L}' = -\frac{6}{\pi}\tilde{\beta}\left(\ln\frac{\tilde{\beta}}{4\pi} + \gamma - \frac{1}{2} - \frac{4\pi^2}{\tilde{f}_r} \right). \tag{58}$$

The lower critical line is automatically obtained by exchanging $\tilde{\beta}$ and $\tilde{L}'$. On the upper critical line (58), $\rho\tilde{\beta}$ and $\rho\tilde{L}'$ are much less than the O (1) term for the small $\tilde{\beta}$ and $\tilde{L}'$ limit. Since LRNO appears between the upper critical line and the lower one (see case (iii) below), the neglect of some terms in obtaining (56) is reasonable. Note that in (56)–(58) the coupling constant term is negligible for La_2CuO_4 since $\tilde{f}_r^{-1} \sim 10^{-11}$.

(iii) The case $\tilde{\beta} = \tilde{L}'$

From equation (44)

$$|\tilde{n}_{N_c}|^2 = \frac{1}{2\tilde{f}_r} - \frac{1}{2\pi\tilde{\beta}^2}\left(\frac{g(\tilde{\beta})}{\pi} + 2s(\tilde{\beta}; \tilde{\beta}, \tilde{\beta}) + h(\tilde{\beta}) \right). \tag{59}$$

Using (48) and (56), we can find the appearance of LRNO for $\tilde{\beta} = \tilde{L}' \gg 1$ and $\tilde{\beta} = \tilde{L}' \ll 1$, respectively. Moreover, through the numerical analysis of (59), we can see that there exists LRNO on the whole $\tilde{\beta} = \tilde{L}'$ line. Thus LRNO appears in the region between the upper and lower critical lines. Of course, this can easily be seen from the fact that the $\tilde{\beta} = \tilde{L}'$ line meets the region where LRNO appears in the infinite-size limit ($\tilde{L}' \to \infty$), that is, they belong to the same region in the phase diagram.

So far we have obtained LRNO and the equations of the critical line in the general case and the limiting cases. For the parameter values of the typical high-temperature superconductor La_2CuO_4, we will plot these equations on logarithmic scales since physical quantities have a very broad range. Figure 4 shows the phase diagram (or the critical lines). The full curves are obtained by solving the exact equation of the critical line (46) numerically. The upper broken curve is the combination of two approximate equations of the critical line in two limiting cases, that is, (55) for the low-temperature and large-size case and (58) for the high-temperature and small-size case. The lower critical lines are obtained automatically by exchanging the coordinates $\ln\tilde{\beta}$ and $\ln\tilde{L}'$ of the upper ones using the exchange symmetry of $\tilde{\beta}$ and $\tilde{L}'$. In the limiting cases approximate critical lines follow the exact one (46) very closely. The asymptotes of the critical lines which exist in the low-temperature and large-size region are $\ln\tilde{\beta}, \ln\tilde{L}' = \frac{1}{2}\ln(\tilde{f}_r/6) \approx 11.23$ (see case (i)). LRNO appears in region I but not in region II. If we see the phase in terms of the spin mass gap σ_c, there is dynamical mass generation in region II but not in region I. The structure of the phase diagram is very different from that of the (1 + 1)-dimensional infinite-N CP^{N-1} model: for fixed $\ln\tilde{L}' \geqslant \ln\tilde{L}'_N$, there is only one critical temperature, as seen from figure 4, while there are

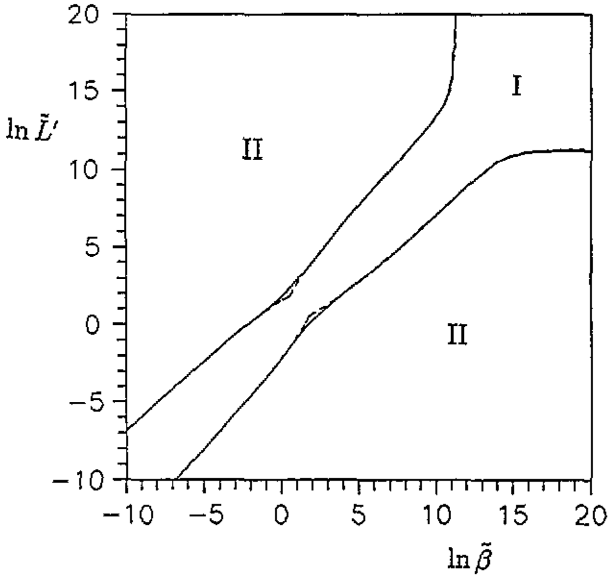

Figure 4. The phase diagram in the $\ln \tilde{\beta}$ and $\ln \tilde{L}'$ space. The full curves are obtained by solving the exact equation of the critical line (46) numerically. The broken curves are obtained by the approximate equations of the critical line: the upper broken curve is the combination of (55) for low-temperature and large-size and (58) for high-temperature and small size. The lower critical line is automatically obtained by the exchange of $\tilde{\beta}$ and $\tilde{L}'$. Note that the approximate critical lines fit the exact ones very well for the limiting cases. LRNO exists in region I but not in region II. The critical lines approach the asymptotic lines $\ln \tilde{\beta} = \ln \tilde{\beta}_N (\approx 11.23)$ and $\ln \tilde{L}' = \ln \tilde{L}'_N = \ln \tilde{\beta}_N$ in the low-temperature and large-size limit. For $\ln \tilde{L}' \geqslant \ln \tilde{L}'_N$, there is only one critical temperature but two critical temperatures exist for $\ln \tilde{L}' < \ln \tilde{L}'_N$.

two critical temperatures for $\ln \tilde{L}' < \ln \tilde{L}'_N$. As we decrease the periodic length L from infinity, the Néel temperature increases along the upper critical line.

The existence of two critical temperatures may be detected by experiments, since the boundary value $\tilde{L}'_N$ corresponds to $L_N \approx 360$ Å for La_2CuO_4. The only critical temperature for $L = L_N$ is $T_c \approx 4100$ K, since $\ln \tilde{\beta}_c \approx 8.2$. Numerical values must be regarded as rough approximations to the real situation. For $L < L_N$, there exist two critical temperatures, T_{c_1} and T_{c_2} ($T_{c_1} < T_{c_2}$), such that LRNO appears only when $T_{c_1} < T < T_{c_2}$. As emphasized in case (i), our results will be valid for real materials only below the melting temperature so that the lattice structure is not destroyed. Since T_{c_2} is higher than the critical temperature for $L = L_N$, it may be difficult to detect it. However, since T_{c_1} increases from zero as L decreases from L_N, it is possible to detect the existence of T_{c_1} such that LRNO vanishes for $0 \leqslant T \leqslant T_{c_1}$. If we consider ordinary antiferromagnetic materials with $\alpha \approx 1$, we will have much larger L_N so that experiments become more accessible.

Figure 5 shows LRNO $\ln \bar{n}_{N_c}$ as a function of $\ln \tilde{\beta}$ for various fixed values of $\ln \tilde{L}'$, which are obtained from (44). Approximate expressions (47), (48) and (56) of LRNO give nearly the same graphs as figure 5. As expected from figure 4, there is only one critical temperature for fixed $\ln \tilde{L}' \geqslant \ln \tilde{L}'_N$, while two critical temperatures exist for fixed $\ln \tilde{L}' < \ln \tilde{L}'_N$. However, the system with $L < a_3$ (lattice constant for the third space dimension) will be physically very unstable.

Finally we discuss the validity of our results in the real situation. The stationary phase

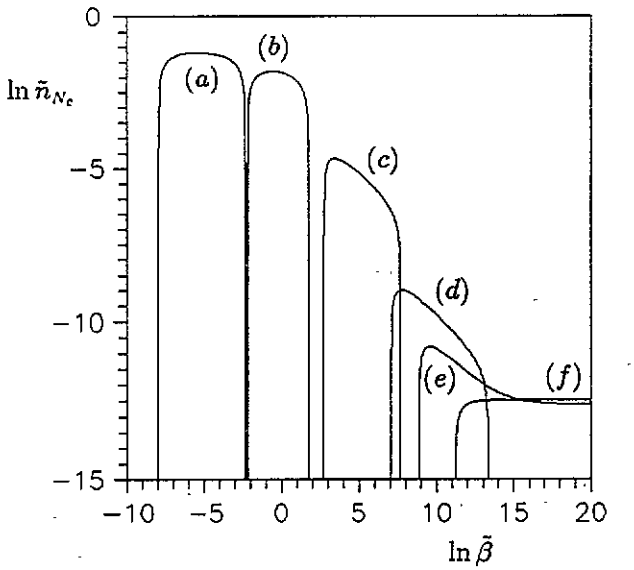

Figure 5. LRNO $\ln \tilde{n}_{N_c}$ as a function of $\ln \tilde{\beta}$ for various values of $\ln \tilde{L}'$; that is, $\ln \tilde{L}' = -5, 0, 5, 10, 12, \infty$ for curves (*a*)–(*f*), respectively. These LRNOs are obtained from the exact equation (44). As expected from figure 4, the cases (*a*)–(*d*) have two critical temperatures between which LRNO appears, while the cases (*e*) and (*f*) have one critical temperature.

method (the saddle-point or the one-loop approximation) adapted to finding LRNO and the equation of the critical line yields the exact result in the large-N limit of the CP^{N-1} model [13]. But for real solids, N is 2 since the spin of electrons is $\frac{1}{2}$. The validity of our results may be suspect in the real situation. However, since the theory is perturbatively non-renormalizable in $d > 1$, it is expected that the quantum fluctuation in higher-loop order is not so large as to change drastically the nature of the phase diagram [3, 5]. In fact the results of the stationary phase approximation for the correlation length as a function of temperature in the (2+1)-dimensional $O(3)$ σ model with infinite size is in good agreement with those of the Monte Carlo simulation [18, 19]. Hence we expect our results to work well for real solids.

5. Conclusions

We have investigated the finite-temperature Néel transition of the anisotropic large-N CP^{N-1} model with one periodic spatial dimension using the effective potential method. We obtained the effective potential as a function of $|n_{N_c}|$ and σ_c, where the first $N-1$ components of the n ($\overline{n}$) field are integrated out and only one component n_N ($\overline{n}_N$) is left, in order to examine the appearance of LRNO. The anisotropic parameter α is absorbed in the redefined periodic length L'. The finite-temperature effective potential is symmetric under exchange of β and L'. Thus LRNO and the equation of the critical line, which are obtained from the equilibrium (stationary phase) conditions of the effective potential, respect the same symmetry.

First the $(1+1)$-dimensional CP^{N-1} model was analysed on toroidal spacetime (i.e. at finite temperature and finite size). We find LRNO and the equation of the critical line. But

LRNO is forbidden by the Mermin–Wagner–Coleman theorem. Thus we conclude that the one-loop approximation fails badly for large but finite N. However, it is exact for infinite N. Hence our results in 1+1 dimensions are valid and LRNO can exist; for the infinite-N CP^{N-1} model, which is not a real physical system, there exists $\tilde{L}_0$ such that our system has two critical temperatures for fixed $\tilde{L}$ satisfying $0 < \tilde{L} < \tilde{L}_0$, while it has no critical temperature for $\tilde{L} \geqslant \tilde{L}_0$. In addition, similar phenomena also occur for the $(2+1)$-dimensional CP^{N-1} model with one periodic spatial dimension [20].

The (3+1)-dimensional CP^{N-1} model is investigated as a model of antiferromagnetism in high-temperature superconductors (e.g. La_2CuO_4), where the anisotropic parameter represents the weak interlayer coupling. We give the third space dimension orthogonal to the planes a periodicity L. LRNO and the equation of the critical line are obtained. The exact formulae are very complicated and thus we find approximate ones in the limiting cases (the large $\tilde{\beta}$ and $\tilde{L}'$ case and the small $\tilde{\beta}$ and $\tilde{L}'$ case) which are very simple and fit the exact ones very well. We use the physical parameters of La_2CuO_4 for comparison with the real situation. The structure of the phase diagram is very different from that of the $(1+1)$-dimensional infinite-N CP^{N-1} model. Our system has only one critical temperature for $L \geqslant L_N$ but two for $L < L_N$; on the other hand, for real solids, $N = 2$. The stationary phase method used here is essentially the same as the large-N approximation of the CP^{N-1} model. However, it is expected that the higher-loop effect is not so large as to change drastically the structure of the phase diagram.

Acknowledgment

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Appendix

For small $\tilde{\beta}$ and $\tilde{L}'$ (i.e. $\tilde{\beta}$ and $\tilde{L}' \ll 1$), we find the series expansion for each term in (44). First consider $g(z)$ for small z, $g(0) = 0$ and

$$\frac{\mathrm{d}g(z)}{\mathrm{d}z} = z \int_0^\infty \mathrm{d}x \frac{1}{\sqrt{x^2+z^2}} \frac{1}{\exp\{\sqrt{x^2+z^2}\}-1}$$
$$= \frac{\pi}{2} + \frac{z}{2} \ln \frac{z}{4\pi} + \frac{\gamma z}{2} + \mathrm{O}(z^2) \qquad \text{(A1)}$$

where we have used (26). Integrating (A1)

$$g(z) = \frac{\pi z}{2} + \frac{z^2}{4} \ln z + \frac{1}{4}\left(\gamma - \frac{1}{2} - \ln(4\pi)\right) z^2 + \mathrm{O}(z^3) \qquad \text{(A2)}$$

for small z. Hence, for small $\tilde{\beta}$

$$\frac{g(\tilde{\beta})}{\pi \tilde{\beta}^2} = \frac{1}{2\tilde{\beta}} + \frac{1}{4\pi} \ln \tilde{\beta} + \frac{1}{4\pi}\left(\gamma - \frac{1}{2} - \ln(4\pi)\right) + \mathrm{O}(\tilde{\beta}). \qquad \text{(A3)}$$

Next consider $s(\tilde{L}'; \tilde{\beta}, \tilde{L}')$ for small $\tilde{\beta}$ and $\tilde{L}'$

$$s(\tilde{L}'; \tilde{\beta}, \tilde{L}') = \sum_{n=1}^{\infty} \ln \left[\frac{1 - \exp\left(-2\pi\rho n\sqrt{1 + (\tilde{\beta}/2\pi n)^2}\right)}{1 - \exp(-2\pi\rho n)} \right]$$

$$= \sum_{n=1}^{\infty} \ln \left[1 - \left\{ -\frac{1}{2}\left(\frac{\tilde{\beta}\tilde{L}'}{2\pi n}\right) + \frac{\tilde{\beta}^3\tilde{L}'}{8(2\pi n)^3} + \frac{1}{8}\left(\frac{\tilde{\beta}\tilde{L}'}{2\pi n}\right)^2 + \cdots \right\} \frac{1}{e^{2\pi\rho n} - 1} \right]$$

$$= \frac{\tilde{\beta}\tilde{L}'}{4\pi} \sum_{n=1}^{\infty} \frac{1}{n} \frac{1}{e^{2\pi\rho n} - 1} - \frac{\tilde{\beta}^3\tilde{L}'}{64\pi^3} \sum_{n=1}^{\infty} \frac{1}{n^3} \frac{1}{e^{2\pi\rho n} - 1}$$

$$- \frac{(\tilde{\beta}\tilde{L}')^2}{32\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^2} \frac{e^{2\pi\rho n}}{(e^{2\pi\rho n} - 1)^2} + \cdots .$$

Then, using (27),

$$\frac{2}{\tilde{\beta}\tilde{L}'} s(\tilde{L}'; \tilde{\beta}, \tilde{L}') \approx \frac{1}{2\pi} \sum_{n=1}^{\infty} \frac{1}{n} \left(\frac{1}{e^{2\pi\rho n} - 1} \right)$$

$$= -\frac{\rho}{24} - \frac{1}{2\pi} \ln \eta(i\rho). \qquad \text{(A4)}$$

Finally, for small $\tilde{L}'$,

$$\frac{h(\tilde{L}')}{\tilde{\beta}\tilde{L}'} = -\frac{1}{2\tilde{\beta}} + \frac{\rho}{24} - \frac{\tilde{L}'^2 \rho}{2880} + \cdots . \qquad \text{(A5)}$$

As a result, for small $\tilde{\beta}$ and $\tilde{L}'$,

$$\frac{g(\tilde{\beta})}{\pi\tilde{\beta}^2} + \frac{2}{\tilde{\beta}\tilde{L}'} s(\tilde{L}'; \tilde{\beta}, \tilde{L}') + \frac{h(\tilde{L}')}{\tilde{\beta}\tilde{L}'} = -\frac{1}{4\pi} \left[2 \ln \eta(i\rho) - \ln \frac{\tilde{\beta}}{4\pi} - \gamma + \frac{1}{2} \right] + \cdots . \qquad \text{(A6)}$$

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